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A POLAR EQUAL-AREA MAP OF THE WORLD

Irving I. Gringorten

Air Force Cambridge Research Laboratories
L. G. Hanscom Field, Massachusetts

4 June 1973

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AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

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The construction of the polar equal-area map is relatively simple for the Northern Hemisphere. In the associated representation of the Southern Hemisphere, however, several difficulties had to be overcome. The entire representation of the globe fits into a square with a loss of only 10.5 percent of the space of the square. Whereas the meridians in the Northern Hemisphere are represented by straight lines radiating from the North Pole, those in the Southern Hemisphere are elliptical in shape, converging from the equator toward the South Pole in four quadrants. The parallels in the Northern Hemisphere are circles centered on the North Pole. The parallels in the Southern Hemisphere follow the curvature of those in the Northern Hemisphere without actually being circles. Beginning with a conformal representation at the North Pole, the distortion increases southward, but is never unacceptable. Except for Antarctica, the continents are not split or divided in this projection.

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In the Design Climatology Branch, Air Force Cambridge Research Laboratories, there frequently arises a need for a global presentation of statistics on the climate. As in most statistical presentations the favored type of map is equal-area. But previous maps have been neglectful of the north and south polar areas. It is desirable to have a projection centered on the North Pole, with habitable land masses realistically grouped around the central point.

The construction of the polar equal-area map is relatively simple for the Northern Hemisphere. In the associated representation of the Southern Hemisphere, however, several difficulties had to be overcome. The entire representation of the globe fits into a square with a loss of only 10.5 percent of the space of the square. Whereas the meridians in the Northern Hemisphere are represented by straight lines radiating from the North Pole, those in the Southern Hemisphere are elliptical in shape, converging from the equator toward the South Pole in four quadrants. The parallels in the Northern Hemisphere are circles centered on the North Pole. The parallels in the Southern Hemisphere follow the curvature of those in the Northern Hemisphere without actually being circles. Beginning with a conformal representation at the North Pole, the distortion increases southward, but is never unacceptable. Except for Antarctica, the continents are not split or divided in this projection.

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1. INTRODUCTION

The construction of a new projection (Figure 1) has been undertaken to improve the mapping of world-wide climatological statistics. As in other projections (Raisz, 1962) the equal-area feature provides the means for comparing the horizontal extent of a feature like a cold climate in one region of the earth to the horizontal extent of the same feature in other regions. But, while most previous equal-area maps have been drawn with the equator as a central straight line, the need has been felt for a top-of-the-world configuration, which would group the large land masses more realistically with respect to each other. In the configuration of Figure 1 the continents are not split except Antarctica. The large land masses of the Southern Hemisphere are shown in three of four quadrants. Beginning with conformality at the North Pole, the distortion increases southward, but it is never felt to be unacceptable.

In the construction of the map the plotting of parallels and meridians in the Northern Hemisphere is quite simple. But the plotting in the Southern Hemisphere was a difficult challenge. A method was devised, however, to obtain as great a degree of accuracy as is permitted by computer technology.

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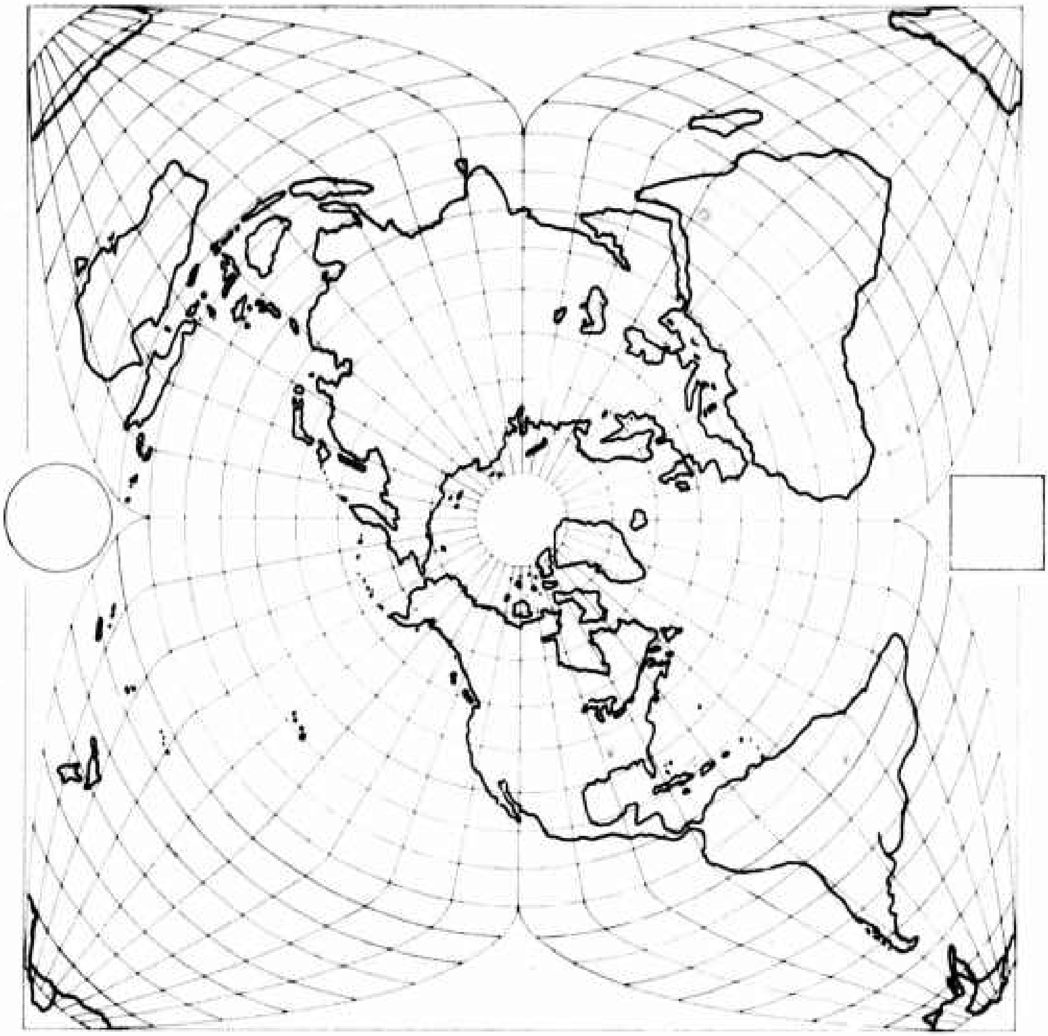

Figure 1. The Polar Equal-Area Map of the World. The circle and square on either side of the map are each one percent of the global area

2. CONSTRUCTION

Suppose the earth's radius is unity. Then the total global area is 4π, and the area of the Northern Hemisphere is 2π. Consequently, for the projection (Figure 2) we make the radius of the circle depicting the Northern Hemisphere (OT_o) equal to $\sqrt{2}$.

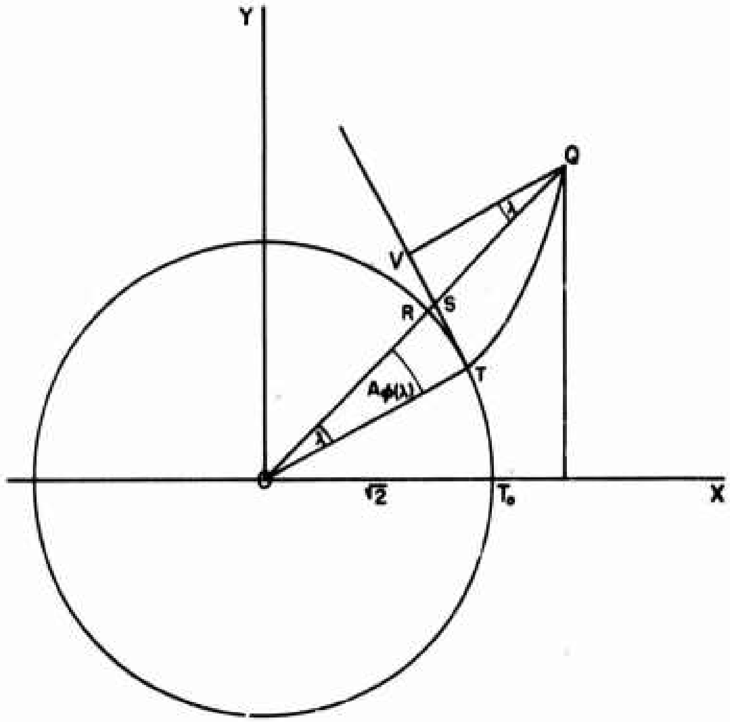

Figure 2. Diagram to Illustrate the Mapping of the Global Surface in Rectangular Coordinates (x, y) with Origin (O) Representing the North Pole. The circle representing the Equator has radius equal to $\sqrt{2}$ Units. The South Pole is represented by Q. The curve (QT) representing the meridian (λ) in the Southern Hemisphere is made elliptical in shape

2.1 Northern Hemisphere

The area of the northern hemispheric surface from the North Pole down to latitude ϕ is $2\pi\,(1-\sin\phi)$. The area of the map circle with radius r corresponding to latitude ϕ is πr^2. Since these two areas are equal, the radius r, corresponding to latitude ϕ in the Northern Hemisphere, is given by

$$r = \sqrt{2(1-\sin\phi)} \quad . \tag{1}$$

Both the northern hemispheric global area and the map area, $A_\phi(\lambda)$, subtended by the longitudinal angle λ, from the North Pole down to latitude ϕ, is

$$A_\phi(\lambda) = \lambda\,(1-\sin\phi) \; . \tag{2}$$

2.2 Southern Hemisphere Meridians

In Figure 2 the point O represents the North Pole, Q represents the South Pole. Line VT is tangent to the equatorial circle at T, which is located on the line OT that makes the longitudinal angle λ, measured clockwise, with OQ. The line VQ is perpendicular to the line VT. The curve QT represents the meridian of longitude λ in the Southern Hemisphere. The map distance

$$OQ = \alpha \tag{3}$$

needs to be fixed as determined below.

At $\phi = 0$, from Eq. (2)

$$\text{area ROT} = \lambda \,. \tag{4}$$

In an equal-area projection the area ROT for the Northern Hemisphere must be equal to the area RQT for the Southern Hemisphere. Thus, referring to Figure 2,

$$\begin{aligned} \text{Area ROT} &= \text{Area RQT} \\ &= \text{Area VQT - Area VQS + Area OTS - Area ROT} \,; \end{aligned}$$

or

$$\lambda = A_c - 1/2\, \overline{VQ}^2 \tan\lambda + \tan\lambda - \lambda \,,$$

where A_c = Area VQT. If

$$F(\lambda) = 2\lambda + (\overline{VQ}^2/2 - 1)\tan\lambda \,, \tag{5}$$

then

$$A_c = F(\lambda) \,. \tag{6}$$

From Figure 2,

$$\overline{QV} = \overline{VQ} = \alpha\cos\lambda - \sqrt{2} \,, \tag{7}$$

$$\overline{VT} = \alpha\sin\lambda \,, \tag{8}$$

where α is defined by Eq. (3).

At this point a decision is made to make the curve QT elliptical in shape, and to set its analytical equation (Figure 3):

$$y'/b = \sqrt{1 - (x'/a)^2} + c + gx' , \tag{9}$$

where the parameters a, b, c, g need to be selected, or determined, to conserve the equal-area characteristic of the map. The X' axis is chosen along, and in the direction of, the straight line VT. The Y' axis is parallel to VQ although not necessarily chosen along VQ.

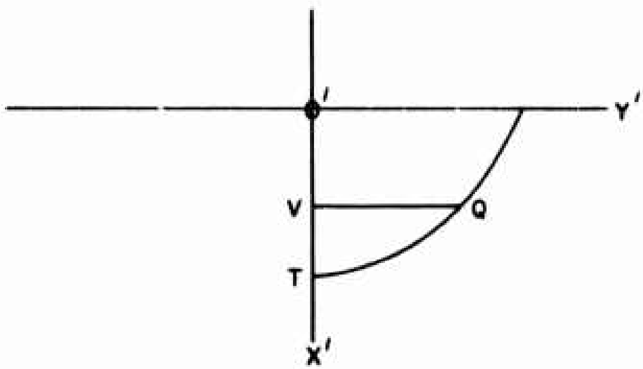

Figure 3. Diagram to Illustrate Eq. (9) of Text

2.2.1 DETERMINATION OF α

Consider the case for $\lambda = \pi/4$ (Figure 4). To make the elliptical curve T_oQ join OT_o as a continuation of OT_o, and to make angle V_oQT_o, in the limit, a right-angle, the area V_oQT_o is made one-fourth of an ellipse with axes:

$$a_o = \overline{V_oT_o} , \tag{10}$$

$$b_o = \overline{V_oQ} . \tag{11}$$

For the remaining terms in Eq. (9)

$$c_o = g_o = 0 . \tag{12}$$

Since the area of an ellipse is π times the product of the semi-axes,

$$\text{Area } V_oQT_o = (A_c)_o = \pi a_o b_o/4 . \tag{13}$$

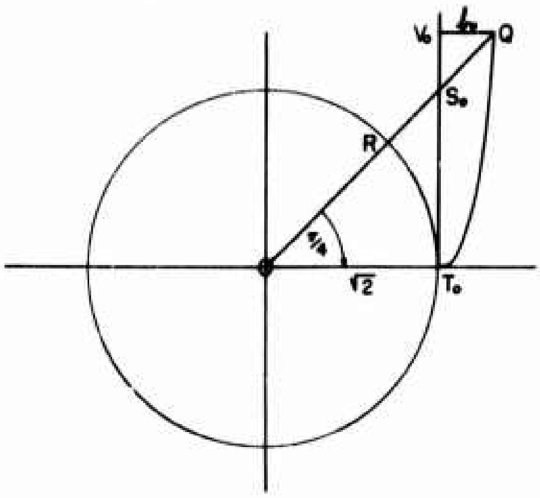

Figure 4. Diagram to Illustrate the Elliptical Curve (T_0Q) of the Outer Meridian of an Octant (QOT_0)

It is also clear (Figure 4) that

$$a_o = b_o + \sqrt{2} \text{ and that}$$

$$\alpha = \overline{OQ} = \sqrt{2}(b_o + \sqrt{2}) \,, \tag{14}$$

which, together with Eq. (5) and Eq. (6) for $\lambda = \pi/4$, yields an expression for b_o which finally gives

$$\alpha = 2 - \left(\frac{\pi}{\pi-2}\right) + \sqrt{\left(\frac{\pi}{\pi-2}\right)^2 + 4}$$

$$= 2.64999707341\cdots \,. \tag{15}$$

With the value of α so established, the lengths of $\overline{VQ}$, $\overline{VT}$ and the area $A_c = F(\lambda)$ are given by Eqs. (5), (6), (7), (8) for all other values of λ (Table 1).

Let Figure 3 be drawn for a given λ ($<\pi/4$), so that

$$a = \overline{O'T} \,.$$

At T, where $x' = a$, $y' = 0$, Eq. (9) gives

$$c = -ga \,. \tag{16}$$

Table 1. The Computed Values of the Significant Parameters in the Determination of the Meridianal Curves (λ) of One Octant in the Southern Hemisphere (see text)

λ	45^{o}	35^{o}	25^{o}	15^{o}	5^{o}	0.01^{o}
λ'	45.0000000	37.06364202	28.00139261	17.73295096	6.203242157	0.01221065989
$F(\lambda)$	0.6764203757	0.7219041892	0.6337181148	0.4314430879	0.1527630859	0.000307802844
$\overline{VT}$	1.873830901	1.519975878	1.119937157	0.6858697120	0.2309624632	0.0004625117387
$\overline{VQ}$	0.4596173384	0.7565369577	0.9874994209	1.145487050	1.225699472	1.235783471
a	1.873830901	1.892277317	1.969040312	1.109939905	0.2061320734	0.001024879313
b	0.4596173385	0.3958947345	0.3682931056	0.2732646802	0.1557473673	0.2791128733
c	0	1.158416665	3.127861067	5.288143332	6.137731418	7.958470779
g	0	-0.6121812348	-1.58852058	-4.76435103	-29.7757225	-7765.276045
λ_1	44.999	34.999	24.999	14.999	4.999	0.009
λ_1'	44.99925982	37.06279349	28.00042732	17.73186150	6.202026583	0.01098243004
$F(\lambda_1)$	0.6764317195	0.7219019019	0.6337030966	0.4314182888	0.1527329876	0.0002770225615
$\overline{VT_1}$	1.873798196	1.519937991	1.119895239	0.6858250368	0.2309163880	0.0004162605652
$\overline{VQ_1}$	0.4596500426	0.7565634859	0.9875189671	1.145499021	1.225703503	1.235783478
a_1	1.873826167	1.892286530	1.969027053	1.109807610	0.2060817917	0.0009975247419
b_1	0.4596084899	0.3958907262	0.3682897681	0.2732498121	0.1557405317	0.2920473696
c_1	0.904102968-4	1.158564672	3.128107789	5.288275099	6.137795374	8.192714016
g_1	-0.48249031-4	-0.6122564704	-1.588656582	-4.76503770	-29.7832978	-8213.043418

Differentiating (9) gives

$$\frac{1}{b}\frac{dy'}{dx'} = g - \frac{1}{a^2}\frac{x'}{\sqrt{1-\frac{x'}{a}^2}} \quad , \tag{17}$$

which implies that the elliptical curve $\overline{TQ}$ meets the tangent $\overline{VT}$, to the circle representing the equator, perpendicularly, and therefore is a continuation of the line OT (Figure 2)

At Q, where $x' = a - \overline{VT}$, $y' = \overline{VQ}$, Eq. (9) gives

$$\frac{\overline{VQ}}{b} = \sqrt{1-(1-\frac{\overline{VT}}{a})^2} - g\,\overline{VT} \,. \tag{18}$$

At Q, set

limiting angle VQT = $2\lambda'$.

It was found generally necessary to make $\lambda' > \lambda$, so that Eq. (9) would have real numbers. From Eq. (17)

$$-\frac{1}{b}\cot 2\lambda' = g - \frac{1}{a}\frac{(1-\frac{\overline{VT}}{a})}{\sqrt{1-(1-\frac{\overline{VT}}{a})^2}} \,. \tag{19}$$

Solving Eqs. (18) and (19) for g and b respectively in terms of a,

$$g = \frac{j}{\overline{VT}} - \frac{1}{b}\frac{\overline{VQ}}{\overline{VT}} \,, \tag{20}$$

$$b = \frac{[\overline{VQ} - \overline{VT}\cot 2\lambda']}{h - ef} \cdot j \,, \tag{21}$$

where

$$e = \overline{VT}/a \,, \tag{22}$$

$$f = 1-e \,, \tag{23}$$

$$h = 1-f^2 \,, \tag{24}$$

$$j = \sqrt{h} \,. \tag{25}$$

For the partial elliptical area VQT (Figure 3)

$$A_c = \int_{a-VT}^{a} y' \, dx' ,$$

which, with Eq. (9), gives

$$A_c = (ab/2) \left[\pi/2 + gah - fj - \sin^{-1} f + 2ce \right] , \tag{26}$$

which, together with Eqs. (5) and (6), becomes a solution for the parameter a, provided a value is selected for λ'.

To make $(\lambda'-\lambda)$ small but large enough for a real curve (TQ), a formula was adopted for λ':

$$\lambda' = \lambda \cdot \exp \left[1/3 \, \lambda^{1/32} \, (\pi/4 - \lambda) \right] . \tag{27}$$

The results of (27) for λ', (26) for a, (20) for g, (21) for f, (16) for c, are as shown (Table 1) corresponding to meridional angles of 0.01° and from 5° to 45° in 10° steps. The values for the parameters have been calculated to 10-figure accuracy. On a desk computer each set of solutions of a, b, c, g, for given λ, were obtained in less than five seconds, even though the parameter a was obtained by trial and error.

With respect to a common set of (X, Y) axes (Figure 2) centered on the North Pole, for given λ,

$$x = (y' + \sqrt{2}) \sin \theta + (x'-a) \cos \theta , \tag{28}$$

$$y = (y' + \sqrt{2}) \cos \theta - (x'-a) \sin \theta , \tag{29}$$

where $\theta = (\pi/4 + \lambda)$.

Thus any meridian (λ) in one octant of the Southern Hemisphere can be plotted as a curve with respect to the (X, Y) axes with origin at the North Pole. Each point (x, y) on the meridional curve (TQ) is obtained for a given x':

$$a-VT \leq x' \leq a .$$

Conversely, from (9) and (28) ,

where

$$k = \cos \theta + bg \sin \theta ,$$
$$m = (bc + \sqrt{2}) \sin \theta - a \cos \theta ,$$

$$p = k^2 + b^2 \sin^2 \theta / a^2 ,$$

$$q = k(m-x),$$

$$r = (m-x)^2 - b^2 \sin^2 \theta .$$

Then x' can be given in terms of x :

$$x' = (-q + \sqrt{q^2 - pr}\,)/p .$$

2.3 Southern Hemisphere Parallels

So far, the solution has yielded equations for the plotting of meridians. Since the shape of the meridians was assigned, the shape of the parallels must follow as a consequence, subject to the constraint that the map is an equal-area projection.

Consider the point F (Figure 5) along the elliptical curve TQ to represent the point of latitude ϕ, longitude λ, in the Southern Hemisphere. If an adjacent curve T_1Q is drawn, such that the angle subtended by T_1T at O is the small increment $\delta\lambda$, then the small incremental area F_1QF must be made equal to the corresponding small incremental area in the Northern Hemisphere, which is given by Eq. (2):

$$A_{\phi}(\delta\lambda) = \delta\lambda\,(1-\sin\phi) . \qquad (31)$$

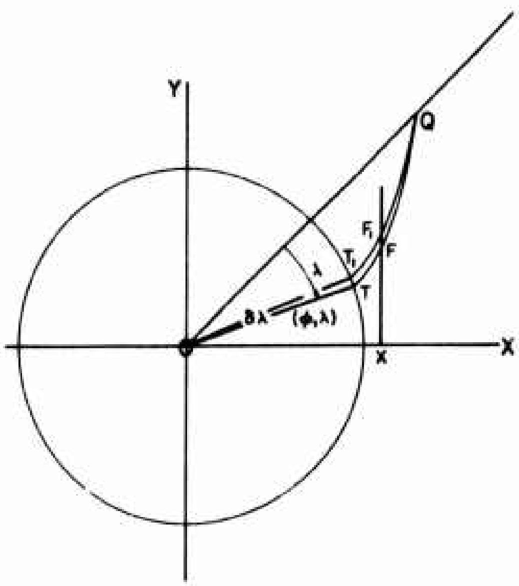

Figure 5. Diagram to Illustrate Two Neighboring Meridians That Differ by a Small Angle ($\delta\lambda$)

With respect to the (X, Y) axes, therefore,

$$\delta\lambda(1-\sin\phi) = \int_{\alpha/\sqrt{2}}^{x} (\delta y)\cdot dx \ , \tag{32}$$

where

$$\delta y = y_1 - y \ ; \tag{33}$$

y_1 is the Y coordinate of F_1 assuming that its X coordinate is the same as for F:

$$x_1 = x \ . \tag{34}$$

For the elliptical curve T_1Q the parameters $[\lambda_1', \ F(\lambda_1), \ \overline{VQ}_1, \ \overline{TV}_1, \ a_1, \ b_1, \ c_1, \ g_1]$ can be determined in the same manner as for the curve TQ, which was done for $\delta\lambda = 0.001^\circ$ (Table 1). The solutions for (x, y), using for (32) the approximate relation

$$\Sigma(y_1-y)\ \Delta x = \delta\lambda\cdot(1-\sin\phi) \tag{35}$$

for $\lambda = 0.01^\circ$ and 5° to 45° in steps of 10°

and for each $\phi = 80^\circ$ to 10° by 10° steps and from 10° to 0° by 1° steps are as shown (Table 2).

The incremental steps on x were made

$$\Delta x = [\alpha/\sqrt{2} - \sqrt{2}\cos(\pi/4-\lambda)]/500 \ , \tag{36}$$

and found to be sufficient for the determination of coordinates (x, y) corresponding to (ϕ, λ) to 3 decimal-point accuracy (Table 2). An exception was made on the boundary meridian $\lambda = 45^\circ$ for $\phi = 80^\circ$ and 75°, for which the denominator in (36) was made 5000, to obtain the coordinates (x, y) with a sufficient degree of accuracy.

Table 2. The Rectangular Coordinates (x, y) Corresponding to (ϕ, λ) for the Plotting of Meridians and Parallels in One Octant of the Southern Hemisphere

λ =	0.01°		5°		15°		25°		35°		45°	
ϕ	x	y	x	y	x	y	x	y	x	y	x	y
85	1.823	1.823	1.830	1.816	1.840	1.800	1.852	1.788	1.862	1.767	1.872	1.706
80	1.771	1.771	1.784	1.757	1.806	1.728	1.829	1.700	1.850	1.663	1.870	1.619
75	1.720	1.720	1.739	1.701	1.773	1.656	1.805	1.610	1.836	1.561	1.864	1.492
70	1.669	1.669	1.694	1.642	1.738	1.582	1.781	1.524	1.821	1.455	1.857	1.364
65	1.617	1.617	1.649	1.586	1.702	1.510	1.756	1.436	1.805	1.351	1.846	1.237
60	1.566	1.566	1.603	1.528	1.669	1.441	1.730	1.348	1.786	1.245	1.833	1.105
55	1.516	1.515	1.558	1.471	1.634	1.371	1.704	1.263	1.766	1.138	1.819	0.984
50	1.465	1.465	1.513	1.415	1.599	1.302	1.677	1.178	1.745	1.036	1.800	0.857
45	1.416	1.416	1.469	1.360	1.564	1.234	1.648	1.093	1.723	0.938	1.779	0.735
40	1.367	1.367	1.425	1.306	1.527	1.165	1.620	1.012	1.698	0.837	1.755	0.618
35	1.318	1.318	1.381	1.251	1.492	1.100	1.590	0.933	1.671	0.740	1.729	0.507
30	1.271	1.271	1.338	1.199	1.457	1.037	1.560	0.855	1.642	0.646	1.697	0.398
25	1.224	1.224	1.295	1.148	1.421	0.974	1.528	0.780	1.611	0.559	1.662	0.297
20	1.177	1.176	1.253	1.097	1.384	0.913	1.495	0.707	1.577	0.473	1.623	0.204
15	1.131	1.131	1.210	1.047	1.348	0.856	1.460	0.639	1.539	0.396	1.578	0.123
10	1.088	1.087	1.169	0.999	1.310	0.801	1.422	0.576	1.497	0.328	1.528	0.059
9	1.079	1.078	1.161	0.990	1.303	0.790	1.414	0.565	1.488	0.316	1.518	0.048
8	1.070	1.070	1.151	0.980	1.295	0.780	1.405	0.553	1.478	0.305	1.507	0.039
7	1.061	1.061	1.143	0.971	1.287	0.770	1.396	0.541	1.469	0.294	1.496	0.030
6	1.053	1.052	1.136	0.962	1.279	0.760	1.389	0.532	1.459	0.284	1.484	0.022
5	1.044	1.044	1.126	0.952	1.270	0.749	1.379	0.521	1.449	0.274	1.473	0.015
4	1.035	1.035	1.118	0.943	1.262	0.740	1.370	0.512	1.438	0.266	1.461	0.010
3	1.026	1.026	1.110	0.935	1.253	0.731	1.361	0.503	1.427	0.259	1.449	0.005
2	1.018	1.017	1.101	0.925	1.244	0.722	1.351	0.495	1.416	0.253	1.437	0.002
1	1.009	1.009	1.093	0.918	1.235	0.714	1.340	0.489	1.404	0.248	1.425	0.001
0	1.000	1.000	1.083	0.909	1.225	0.707	1.329	0.484	1.393	0.246	1.413	0.000

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Raisz, Erwin (1962) Principles of Cartography, McGraw-Hill, New York, 315 pp.